First Order Differential Equations

Seperable Equations A differential equation is called seperable if it is of the form

$$g(y)y' = f(x)$$

An equation is seperable if we can isolate all y terms on one side of the equation and all x terms on the other side. Equations of this type can be solved by integrating each side of the equation with respect to the appropriate variable.

Examples

1. $y' = yx$

   This equation is seperable, as can be seen after dividing by y. This gives $\frac{y'}{y} = x$. Integrating both sides gives $\ln y = x + C \implies y = e^{x+C} = Ce^x$. When we divided by y, we tacitly assumed that $y \neq 0$. We must therefore check if $y = 0$ solves the differential equation. The soltuions are then $y = 0$ and $y = Ce^x$.

2. $2xy^2 - x^4y' = 0$

   We can rearrange this equation to give $\frac{2}{x^3} = \frac{y'}{y^2}$. This is seperable, and the solution is revealed by integrating. $\frac{-1}{x^2} + C = \frac{-1}{y} \implies y = \frac{x^2}{1+Cx^2}$.

First Order Linear Equations These differential equations take the general form

$$y' + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are functions of x only. The following are examples of linear equations.

1. $y' + x^2y = 0$
2. $y' + \cos(x)\ y = x^2$
3. $y' + \frac{y}{1-x} = e^x$

The following equations would not qualify as linear.

1. $(y')^2 - \sin(x)\ y = 0$
2. $y' + \frac{x^2}{y} = 2x$
3. $y' + e^xy = y^2$

To solve these equations, we use the integrating factor $\mu = e^{\int p(x)\,dx}$. With this integrating factor, the solution can then be written as $y = \frac{1}{\mu}\int \mu\, q(x)\, dx$.

Examples

1. $y' + \frac{y}{x} = 2e^{x^2}$

   In this case, $p(x) = \frac{1}{x}$ and $\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$. Using our above equation for y gives the solution $y = \frac{1}{x}\int 2xe^{x^2}\,dx = \frac{1}{x}(e^{x^2} + C)$

2. $y' + y\cos x = \cos x$

   In this case, $p(x) = \cos x$ and $\mu = e^{\int \cos x\,dx} = e^{\sin x}$. Again, applying the solution equation gives $y = \frac{1}{e^{\sin x}}\int \cos x\; e^{\sin x}\,dx = e^{-\sin x}(e^{\sin x} + C) = 1 + Ce^{-\sin x}$

Exact Equations An equation of the form

$$M\;dx + N\;dy = 0$$

with M and N functions of x and y, is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

To solve an exact equation, we follow these steps:

1. Our solution will be $F(x, y) = \Psi(y) + \int M\;dx = C$, where $\Psi(y)$ is a function entirely of y to be found later.
2. Calculate the integral $\int M\;dx$.
3. Take the derivative of $F(x, y)$ with respect to y. Set this equal to N and solve for $\Psi'(y)$. $\Psi'(y) = N - \frac{\partial \int M\;dx}{\partial y}$.
4. Find $\Psi(y)$ by integrating $\Psi'(y)$ with respect to y. $\Psi(y) = \int \Psi'(y)\;dy$.
5. Plug $\Psi(y)$ into $F(x, y)$ to obtain the solution.

Examples

1. $2xy\;dx + (x^2 + 2y)\;dy = 0$

   Here $M = 2xy$ and $N = x^2 + 2y$. We see the equation is exact since $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$. $F(x, y) = \int 2xy\;dx + \Psi(y) = x^2y + \Psi(y)$. Now we solve for $\Psi(y)$. $\Psi'(y) = N - \frac{\partial(x^2y)}{\partial y} = (x^2 + 2y) - x^2 \implies \Psi'(y) = 2y$. Integrating we see that $\Psi(y) = y^2$. Our solution is then $x^2y + y^2 = c$.

2. $(2xy - 9x^2)\ dx + (2y + x^2 + 1)\ dy = 0$

Here $M = 2xy - 9x^2$ and $N = 2y + x^2 + 1$. We see the equation is exact since $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$. $F(x,y) = \int 2xy - 9x^2\ dx + \Psi(y) = x^2y - 3x^3 + \Psi(y)$. Next, solve for $\Psi(y)$. $\Psi'(y) = N - \frac{\partial(x^2y-3x^3)}{\partial y} = (2y+x^2+1) - x^2 = 2y+1$. Integrate this to see that $\Psi(y) = y^2 + y$. The solution is then $F(x,y) = x^2y - 3x^3 + y^2 + y = C$.

Making Equations Exact Ocassionally, one will encounter an equation of the form

$$M\ dx + N\ dy = 0$$

that does not meet the criterion for exactness. In certain situations, we can find an appropriate integrating factor which will transform this into an exact equation.

Case 1 Integrating factors of x only: If the quantity $p(x) = \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function with no occurances of y, then $\mu = e^{\int p(x)\ dx}$ is an integrating factor for the differential equation.

Case 2 Integrating factors of y only: If the quantity $p(y) = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function with no occurances of x, then $\mu = e^{\int p(y)\ dy}$ is an integrating factor for the differential equation.

When the integrating factor μ exists, one may multiply the differential equation by μ to created an exact equation.

Examples

1. $(y^2(x^2+1) + xy)\ dx + (2xy + 1)\ dy = 0$

$\frac{\partial M}{\partial y} = 2y(x^2+1) + x$, and $\frac{\partial N}{\partial x} = 2y$. As we can see, this equation is not exact. We will search for an integrating factor. $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y(x^2+1)+x-2y}{2yx+1} = \frac{2yx^2+x}{2yx+1} = x$. This a function entirely of x so that $\mu = e^{\int x\ dx} = e^{\frac{x^2}{x}}$ will be an integrating factor.

Multiply the initial equation by μ to give $(e^{\frac{x^2}{2}}y^2(x^2+1) + e^{\frac{x^2}{2}}xy)\ dx + (2e^{\frac{x^2}{2}}xy + e^{\frac{x^2}{2}})\ dy = 0$. Now $\frac{\partial M}{\partial y} = 2x^2e^{\frac{x^2}{2}}y + 2ye^{\frac{x^2}{2}} + xe^{\frac{x^2}{2}} = \frac{\partial N}{\partial x}$ so that the equation is now exact and can be solved via the methods previously discussed.

2. $(x^2y + 2y^2\sin x)\ dx + (\frac{2}{3}x^3 - 6y\cos x)\ dy = 0$

The equation is not exact since $\frac{\partial M}{\partial y} = x^2 + 4y\sin x$, and $\frac{\partial N}{\partial x} = 2x^3 + 6y\sin x$. Now attempt to find an integrating factor. $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{2x^2 + 6y\sin x - x^2 - 4y\sin x}{x^2y + 2y^2\sin x} = \dfrac{x^2 + 2y\sin x}{x^2y + 2y^2\sin x} = \frac{1}{y}$.

This is a function entirely of y so the equation has an integrating factor of the form $e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$.

Multiply the initial equation by y to give $(x^2y^2 + 2y^3 \sin x)\ dx + (\frac{2}{3}x^3y - 6y^2 \cos x)\ dy = 0$. Now $\frac{\partial M}{\partial y} = 2x^2y + 6y^2 \sin x = \frac{\partial N}{\partial x}$. As we can see, this equation is now exact and can be solved accordingly.